# Greedy Algorithms: The Fractional Knapsack

Version of November 5, 2014

## Outline

- Introduction
- The Knapsack problem.
- A greedy algorithm for the fractional knapsack problem
- Correctness

# Introduction to Greedy Algorithm

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- Final output is an optimal solution.
- Greedy algorithms don't always yield optimal solutions but, when they do, they're usually the simplest and most efficient algorithms available.

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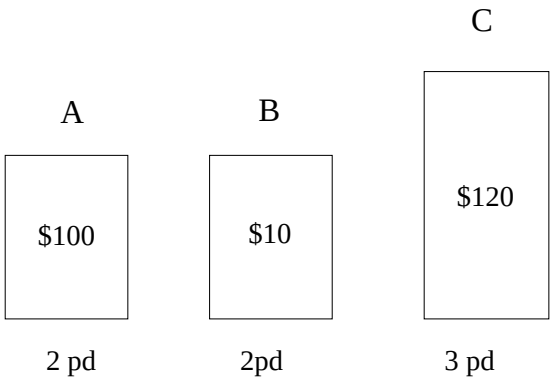
# The Knapsack Problem...

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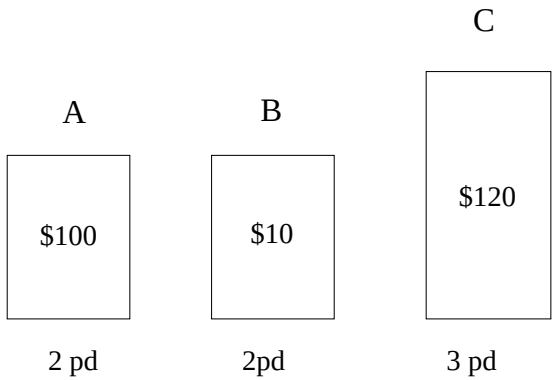


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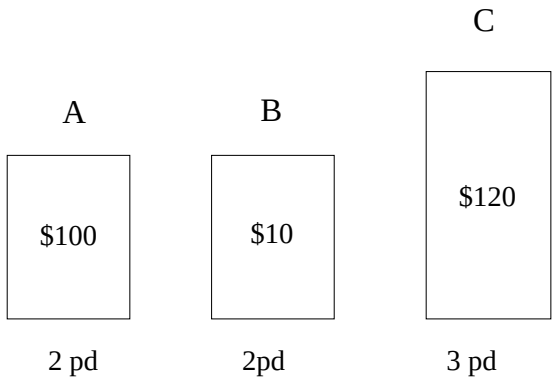


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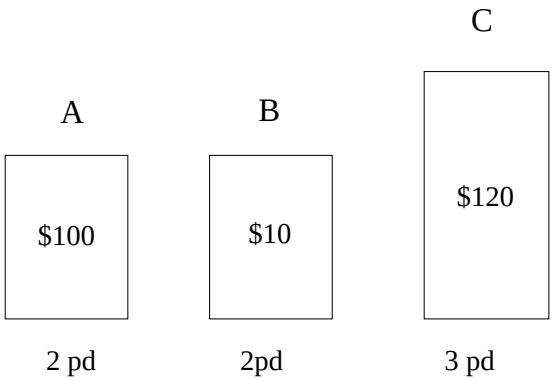


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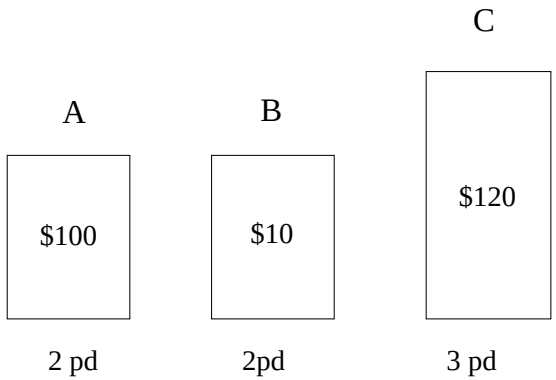


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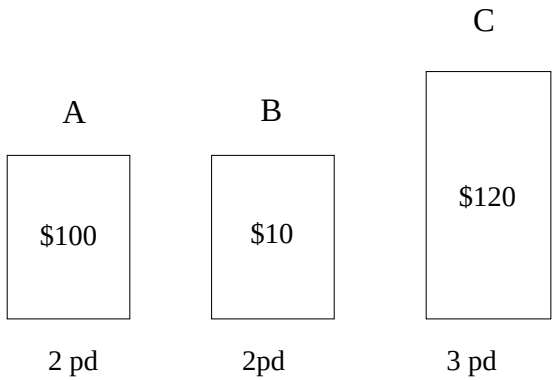


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| | |
|---|---|
| 3 pd<br>C<br>$120 | |

# The Fractional Knapsack Problem: Formal Definition

- Given $K$ and a set of $n$ items:

| weight | $w_1$ | $w_2$ | $\ldots$ | $w_n$ |
|---|---|---|---|---|
| value | $v_1$ | $v_2$ | $\ldots$ | $v_n$ |

- Find: $0 \leq x_i \leq 1$, $i = 1, 2, \ldots, n$ such that

$$\sum_{i=1}^{n} x_i w_i \leq K$$

and the following is maximized:

$$\sum_{i=1}^{n} x_i v_i$$

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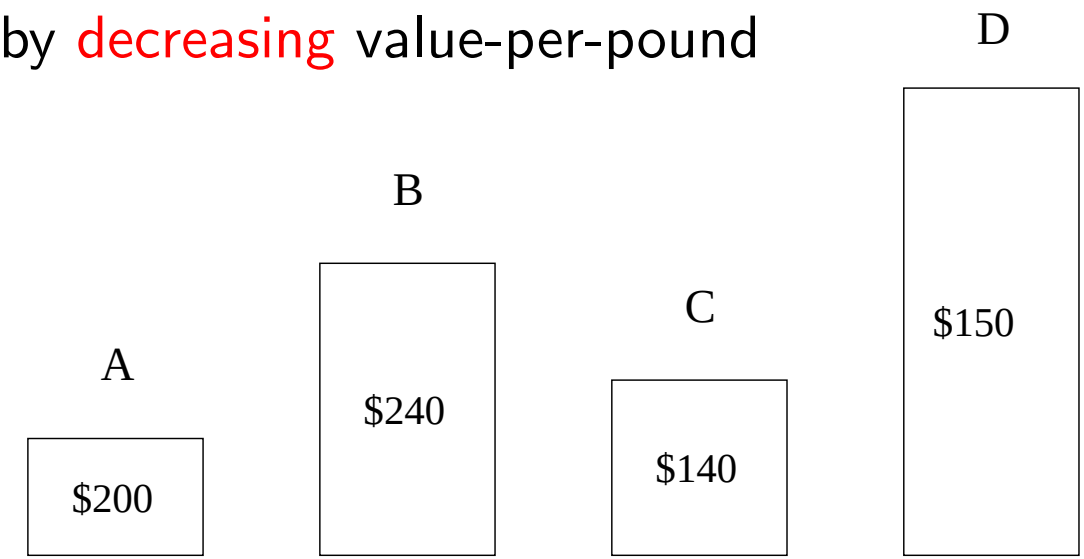

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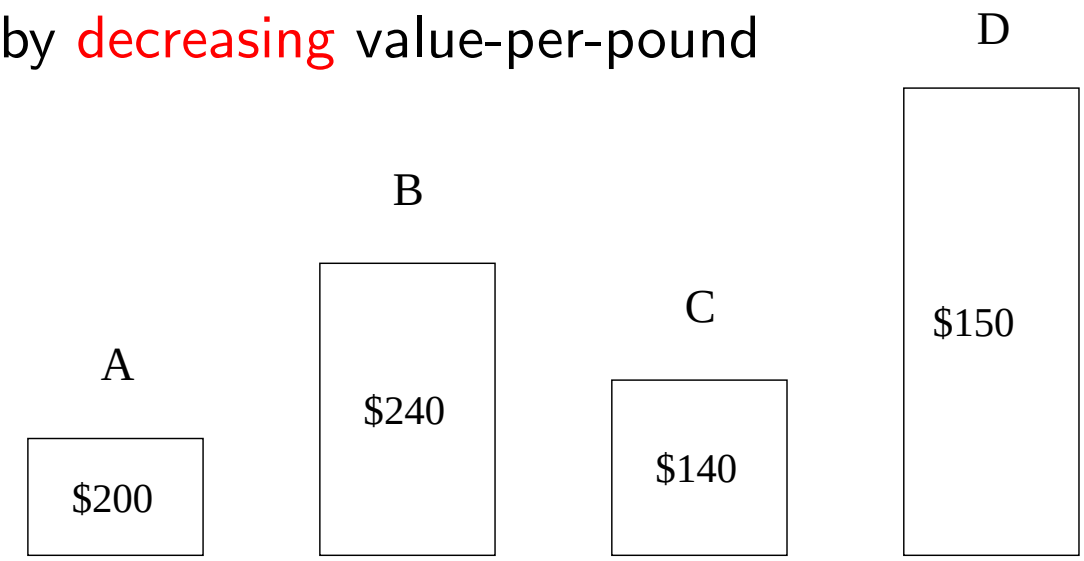


If knapsack holds $K = 5$ pd, solution is:

| | | |
|---|---|---|
| 1 | pd | A |
| 3 | pd | B |
| 1 | pd | C |

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Running time: $O(n \log n)$.

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- We claim that the total value for this set of items is the optimal value.

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**Therefore $G$ is also optimal!**

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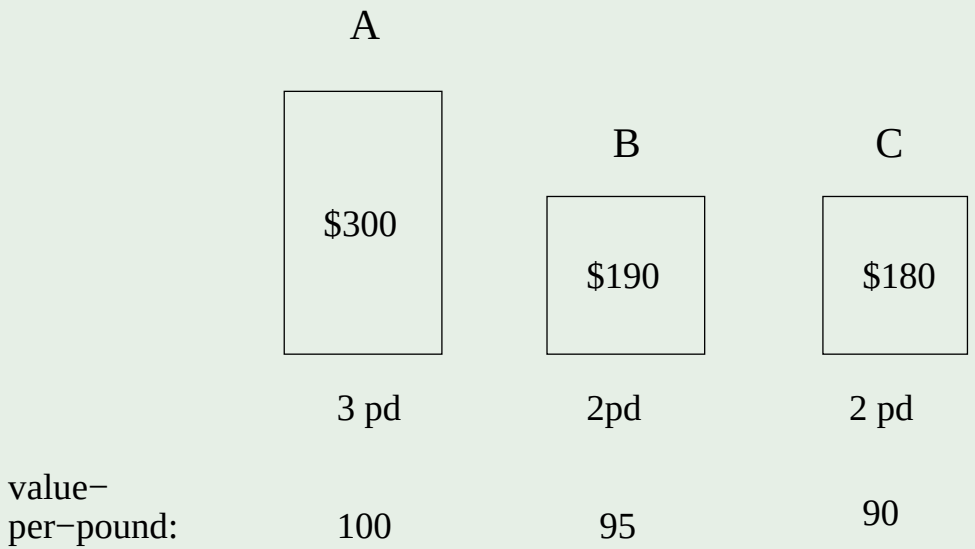
The 0-1 Knapsack Problem does not have a greedy solution!

## Example

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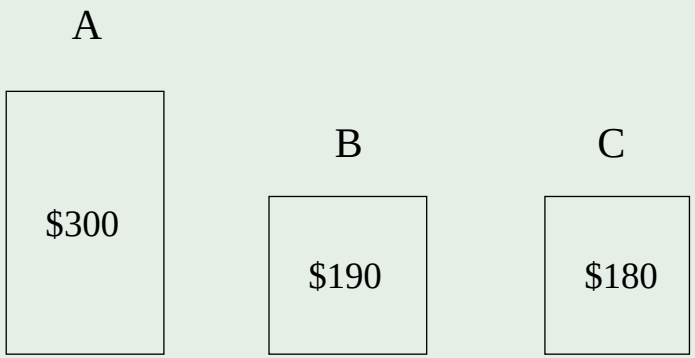


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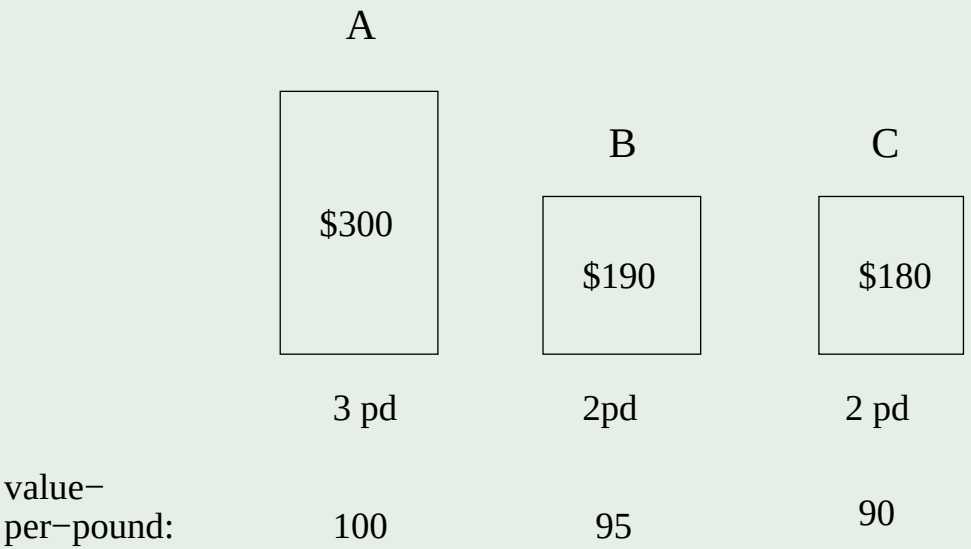


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## Question

Suppose we tried to prove the greedy algorithm for 0-1 knapsack problem **does** construct an optimal solution.

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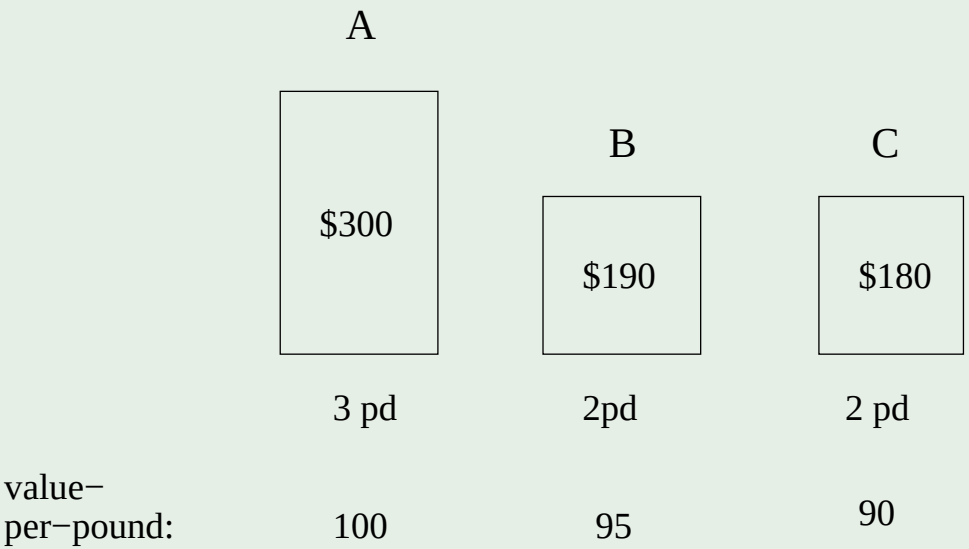
The 0-1 Knapsack Problem does not have a greedy solution!

## Example




$K = 4$. Solution is item B + item C

## Question

Suppose we tried to prove the greedy algorithm for 0-1 knapsack problem **does** construct an optimal solution. If we follow exactly the same argument as in the fractional knapsack problem where does the proof fail?